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LETTER TO THE EDITOR

Sum rules for quantum billiards

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Abstract. We write explicit integral expressions for sums of inverse powers of the eigenvalues of the Laplacian with Dirichlet boundary conditions in a simply connected bounded two-dimensional domain.

In the course of an investigation of the spectrum of the Laplacian in two-dimensional systems, in particular triangles which will be the example discussed below, we came across a simple observation which yields an expression for sums of eigenvalues to negative integral powers. Although the derivation is easy, we were unable to trace it in the literature.

Consider a simply connected bounded domain T in the Euclidean plane in which we wish to find the spectrum of minus the Laplace operator with Dirichlet boundary conditions, i.e. vanishing of the eigenfunctions on the boundary ∂T. This ensures that the ground state has a positive eigenvalue. Call these discrete eigenvalues $E_1 < E_2 \leqslant E_3 \ldots$, and let

$$\zeta(T; s) = \sum_{p=1}^{\infty} 1/E_p^s. \tag{1}$$

From the Riemann mapping theorem, there exists a conformal one-to-one map $t \leftrightarrow z$, with t inside T and z in the upper half-plane $\operatorname{Im} z > 0$, in such a way that the boundary of T be mapped on the real axis. This is realised by an analytic function $t(z)$ univalent in the upper half-plane, which can therefore be inverted in the form $z(t)$, $t \in T$. Let t, t_0 lie inside T, with z, z_0 their images. Set

$$G(t, t_0) = \frac{1}{2\pi} \ln\left|\frac{z-\bar{z}_0}{z-z_0}\right|. \tag{2}$$

This is a function which vanishes on ∂T, is harmonic in the variable t in $T-\{t_0\}$, behaves as $-(2\pi)^{-1}\ln|t-t_0|$ as $t \to t_0$ and is symmetric in the interchange $t \leftrightarrow t_0$. In other words

$$-\Delta_t G(t, t_0) - \delta(t, t_0) \qquad G(t, t_0)|_{t \in \delta T} = 0. \tag{3}$$

Consequently with $t_{n+1} \equiv t_1$ and $z_{n+1} \equiv z_1$, $n \geqslant z$:

$$\zeta(T, n) = \sum_{p=1}^{\infty} \frac{1}{E_p^n} = \int_{t_i \in T} \prod_{i=1}^{n} [\mathrm{d}^2 z_i G(t_i, t_{i+1})] = \int_{\operatorname{Im} z_i > 0} \prod_{i=1}^{n} \left(\frac{\mathrm{d}^2 z_i}{2\pi |p(z_i)|^2} \ln \left| \frac{z_i - \bar{z}_{i+1}}{z_i - z_{i+1}} \right| \right) \tag{4}$$

with

$$p(z) = (\mathrm{d}z/\mathrm{d}t)(z). \tag{5}$$

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When $p(z)$ is known as a function of z, this provides a sequence of explicit formulae. Of course $\zeta(T, s)$ has a simple pole at $s=1$.

Weyl's asymptotic formula yields the residue of this pole in terms of the area A of T, and following an idea of Voros one can obtain the finite part as a single integral in the form

$$\begin{aligned} &\zeta(T; s) = (A/4\pi)[(s-1)^{-1} + g + \mathrm{O}(s-1)] \\ &g = 2\gamma + A^{-1} \int_{\mathrm{Im}\, z>0} \frac{\mathrm{d}^2 z}{|p(z)|^2} \ln \left|\frac{\mathrm{Im}\, z}{p(z)}\right|^2 \\ &A = \text{area of } T = \int_{\mathrm{Im}\, z>0} \frac{\mathrm{d}^2 z}{|p(z)|^2} \end{aligned} \tag{6}$$

where $\gamma =$ Euler's constant $= 0.577\,21\ldots$

To prove (6) set, for $1 < s < 2$,

$$\begin{aligned} &\zeta(T, s) = \frac{1}{\Gamma(2-s)\Gamma(s)} \int_0^\infty \mathrm{d}\lambda\, \lambda^{1-s} \frac{\mathrm{d}R(\lambda)}{\mathrm{d}\lambda} \\ &\frac{\mathrm{d}R(\lambda)}{\mathrm{d}\lambda} = \sum_{p=1}^{\infty} \frac{1}{(\lambda + E_p)^2}. \end{aligned} \tag{7}$$

The sum for $\mathrm{d}R(\lambda)/\mathrm{d}\lambda$ converges for $(-\lambda)$ outside the spectrum, and $R(\lambda)$ is chosen conveniently as

$$R(\lambda) = \int_0^\lambda \mathrm{d}\mu \frac{\mathrm{d}R(\mu)}{\mathrm{d}\mu} = \int_{t\in T} \mathrm{d}^2 t \lim_{t'\to t} [G(t, t') - G(\lambda|t, t')]. \tag{8}$$

Here $G(\lambda|t, t')$, with $G(0|t, t') \equiv G(t, t')$, is the generalisation of (2) fulfilling

$$(-\Delta + \lambda) G(\lambda|t, t') = \delta(t, t') \qquad G(\lambda|t, t')|_{t\in\partial T} = 0. \tag{9}$$

As $\lambda \to \infty$, the Balian-Bloch (1970) representation of the Green function enables one to replace in (8), up to exponentially small terms, $G(\lambda|t, t')$ by $G_0(\lambda|t, t')$, the free Green function

$$G_0(\lambda|t, t') = \frac{1}{2\pi} k_0(\lambda^{1/2}|t-t'|) \underset{t\to t'}{\sim} \frac{1}{2\pi}\left(\ln \frac{2}{\lambda^{1/2}|t-t'|} - \gamma\right) + \ldots. \tag{10}$$

K_0 is the modified Bessel function.

Returning to equation (7), using the behaviour

$$\begin{aligned} &\lambda \sim 0 \qquad R(\lambda) = \lambda \zeta(T, 2) + \mathrm{O}(\lambda) \\ &\lambda \sim \infty \qquad R(\lambda) = \frac{A}{4\pi} \ln \lambda + \frac{A\gamma}{2\pi} + \int_{t\in T} \mathrm{d}^2 t \lim_{t'\to t} \left(G(t, t') - \frac{1}{2\pi} \ln \frac{2}{|t-t'|}\right) + \mathrm{O}(1) \end{aligned} \tag{11}$$

we find by splitting the integral for $\zeta(T, s)$ in $[0, \Lambda]$ and $[\Lambda, \infty]$, where $\Lambda \gg 1$, that

$$\begin{aligned} \zeta(T, 1+\varepsilon) &= R(\Lambda) + (A/4\pi\varepsilon\Lambda^{\varepsilon}) + \mathrm{O}(\varepsilon) \\ &= \frac{A}{4\pi}\left(\frac{1}{\varepsilon} + 2\gamma + \int_{\mathrm{Im}\, z>0} \frac{\mathrm{d}^2 z}{|p(z)|^2} \ln \left|\frac{\mathrm{Im}\, z}{p(z)}\right|^2 + \mathrm{O}(\varepsilon)\right) \end{aligned} \tag{12}$$

which is the result stated in (6).

As an example consider the Schwarz mapping of a triangle on the upper half-plane, defined by

$$t=\int_0^z \mathrm{d}x\, x^{\alpha_0-1}(1-x)^{\alpha_1-1} \qquad \operatorname{Im} z>0 \tag{13}$$

with the principal determination of the powers, as x tends to the real axis between 0 and 1. The triangle $T_{(\alpha_0,\alpha_1,\alpha_\infty)}$ has vertices t_0, t_1, t_∞, corresponding to $(0;1;\infty)$ in the z plane, and corresponding angles given by

$$\begin{array}{lll} z & t & \text{angle} \\ 0 & t_0=0 & \pi\alpha_0 \\ 1 & t_1=\Gamma(\alpha_0)\Gamma(\alpha_1)/\Gamma(\alpha_0+\alpha_1) & \pi\alpha_1 \\ \infty & t_\infty=\mathrm{e}^{\mathrm{i}\pi\alpha_0}\Gamma(\alpha_0)\Gamma(\alpha_\infty)/\Gamma(\alpha_0+\alpha_\infty) & \pi\alpha_\infty \end{array} \tag{14}$$

with

$$0<\alpha_0,\alpha_1,\alpha_\infty<1 \qquad \alpha_0+\alpha_1+\alpha_\infty=1. \tag{15}$$

An overall scale has been chosen for convenience. The function p, to be used in the sum rules, is

$$p(z)=\mathrm{d}z/\mathrm{d}t=z^{1-\alpha_0}(1-z)^{1-\alpha_1} \tag{16}$$

and the area A of the triangle is

$$A=\int\frac{\mathrm{d}^2z}{|p(z)|^2}=\frac{\pi}{2}\frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_\infty)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1+\alpha_\infty)}. \tag{17}$$

In the scale given by (17) the constant g of equation (6) can explicitly be computed in terms of the logarithmic derivative $\psi(\alpha)$ of the Euler Γ function as

$$g(\alpha_0,\alpha_1,\alpha_\infty)=\gamma-2\ln 2+\sum_{i=0;1;\infty}[\alpha_i\psi(\alpha_i)-(1-\alpha_i)\psi(1-\alpha_i)]. \tag{18}$$

The case of integrable triangles is a test of any evaluation of these formulae. In general, when α_0 and α_1 are rational, and if q denotes their least common multiple, equation (16) turns into an algebraic differential equation (i.e. the vanishing of a polynomial in z and $\mathrm{d}z/\mathrm{d}t$) of the form $(\mathrm{d}z/\mathrm{d}t)^q=z^{q(1-\alpha_0)}(1-z)^{q(1-\alpha_1)}$.

Integrability corresponds to the cases $(\alpha_0,\alpha_1,\alpha_\infty)=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$ and $(\frac{1}{6},\frac{1}{3},\frac{1}{2})$, up to permutations. For instance, for an equilateral triangle (Itzykson and Luck 1986)

$$(4\pi^2/3^{3/2}A)]^s\zeta(\tau_{(1/3,1/3,1/3)},s)=\zeta(s)L(s)-\zeta(2s) \tag{19}$$

where Riemann's ζ function appears on the RHS

$$\zeta(s)=\sum_1^\infty\frac{1}{n^s}=\prod_{p\text{ prime}}\frac{1}{(1-p^{-s})} \tag{20}$$

and $L(s)$ is a Dirichlet series

$$\begin{aligned} L(s)&=\sum_{n=0}^\infty\left(\frac{1}{(3n+1)^s}-\frac{1}{(3n+2)^s}\right)\\ &=\prod_{p\text{ prime}\equiv 1\,(\mathrm{mod}\,3)}\frac{1}{(1-p^{-s})}\prod_{r\text{ prime}\equiv 2\,(\mathrm{mod}\,3)}\frac{1}{(1+r^{-s})} \end{aligned} \tag{21}$$

$$L(1)=\frac{\pi}{3^{3/2}} \qquad L(0)=\tfrac{1}{3}$$

$$L(n+1)=[(-3)^{-(n+1)}/n!][\psi^{(n)}(\tfrac{1}{3})-\psi^{(n)}(\tfrac{2}{3})] \qquad n\geqslant 1.$$

Combining (19)-(21) with the sum rule for $n=1$, i.e. the one giving the finite part g through equation (18), we derive the value of the logarithmic derivative of the above Dirichlet series at $s=1$:

$$\frac{L'(1)}{L(1)}=\gamma+\ln\left(\frac{(2\pi)^4}{3^{3/2}\Gamma(\frac{1}{3})^6}\right). \tag{22}$$

This can be checked directly using the symmetry property

$$2\sin\tfrac{1}{2}\pi s\Gamma(s)L(s)=3^{1/2}(\tfrac{2}{3}\pi)^s L(1-s) \tag{23}$$

and the product expansion of Γ functions.

The energy E_1 of the ground state is given by the monotonic limit of estimates (lower bounds)

$$E_1=\lim_{n\to\infty}E_1^{(n)} \qquad E_1^{(n)}=\zeta(T,n)^{-1/n}. \tag{24}$$

We have tested the convergence of the sequence (24) on the example of isosceles triangles ($\alpha_0=\alpha_1=\alpha$, $\alpha_\infty=1-2\alpha$), keeping a fixed area. The integrals $\zeta(T,n)$ have been evaluated for $n\leqslant 4$ using a grid of K^{2n} points in bipolar coordinates, and extrapolation with respect to K through Neville techniques. The convergence of the sequence (24) is satisfactory, even for values of n as small as four. The estimate $E_1^{(4)}$

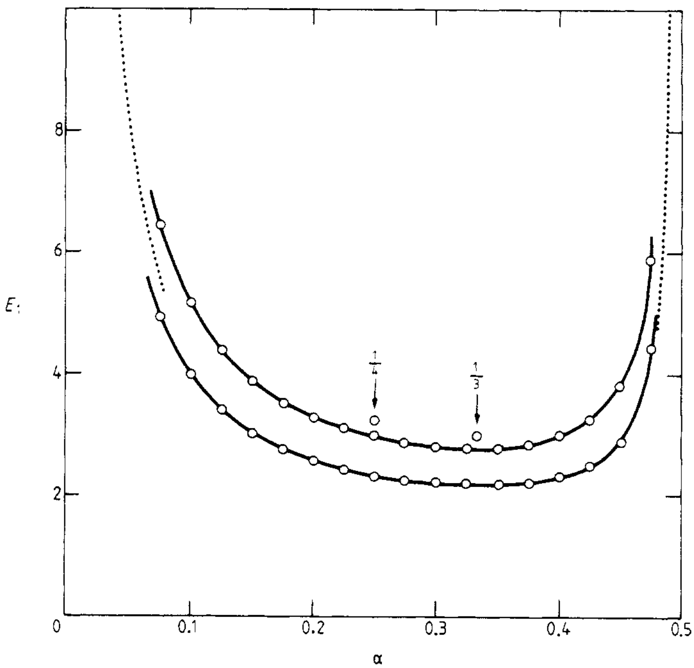

Figure 1. Plot of the estimates $E_1^{(2)}$ (lower full curve) and $E_1^{(3)}$ (upper full curve) of the ground state energy of isosceles triangles, in units $(2\pi)^2/(3\sqrt{3}A)$, against the parameter α, compared with two exact results (arrows) and asymptotic expressions (broken curves).

is very close (less than 1% relative accuracy) to the exactly known results in two particular integrable cases:

$$\begin{aligned} &\alpha = \tfrac{1}{3} \text{ (equilateral triangle)} && E_1 = (2\pi)^2/\sqrt{3}A \\ &\alpha = \tfrac{1}{4} \text{ (rectangular isoceles triangle)} && E_1 = \tfrac{5}{8}(2\pi)^2/A. \end{aligned} \tag{25}$$

The computation of $E_1^{(4)}$ needs much more computer time for less symmetric geometries ($\alpha \to 0$ or $\frac{1}{2}$). Figure 1 shows plots of the estimates $E_1^{(2)}$ and $E_1^{(3)}$ against the parameter α, together with the exact results (25) and the asymptotic expressions

$$E_1 \underset{\alpha\to 0}{\sim} \pi/\alpha A \qquad E_1 \underset{\alpha\to\frac{1}{2}}{\sim} \pi/[2(1-2\alpha)A]. \tag{26}$$

In these limiting situations the ground state wavefunction becomes concentrated in the region of largest breadth. The data confirm our intuition that, for fixed area, the energy is minimal in the most symmetric situation of an equilateral triangle.

In a recent preprint Berry (1985) has also made use of these sum rules in conjunction with asymptotic estimates to approximate the ground state energy.

N Balazs and A Voros have developed similar ideas in an analogous problem. We thank them for stimulating conversations. In particular, the proof of equation (6) is based on some unpublished work of A Voros.

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